An Analytic Solution for Entry into Planetary Atmospheres

S. J. CITRON* AND T. C. MEIR†
Purdue University, Lafayette, Ind.

Approximate closed form solutions for inclination angle and density or altitude as a function of velocity are obtained for a body entering a planetary atmosphere. The method of solution used assumes that the atmospheric density may be expanded in a power series in terms of the logarithm of the velocity. Termination of the series after the first three terms is shown to be equivalent to a result obtained by Loh on the basis of observations that a certain combination of the entry variables may be considered constant. The approximate solution obtained here is compared with the exact solution, and other approximate solutions obtained by Loh and Allen and Eggers, over wide regions of lift to drag ratio, entry velocity, and entry inclination angle. Good agreement is found over all regions except those corresponding to certain skipping-type trajectories. For this critical case, the solution of Loh is also inaccurate. In general, the solution obtained yields the same accuracy as the "second order theory" of Loh, although it is somewhat simpler to apply.

Nomenclature

A	= entry vehicle reference area
B	= ballistic coefficient, mg/C_DA
C	= Euler's const, 0.57721...
C_L	= lift coefficient
C_D	= drag coefficient
D	= drag
e	= base of the natural logarithm, 2.71828...
$E_i^*(\bar{Z})$	= exponential integral $\int_{-\infty}^{\bar{z}} \frac{e}{\alpha}^{\alpha} d\alpha$
f	= nondimensional density, ρ/ρ_0
g	= acceleration of gravity
h	= altitude
$\bar{I}$	= grouping of coefficients, $(\beta/r)(B\cos\gamma/g\ \rho_0)^2$
I	= grouping of coefficients, $(\beta/r)(B\cos\gamma_e/g\ \rho_0)^2$
$\bar{J}$	= grouping of coefficients, $\frac{1}{2}(L/D)(\bar{I}\beta r)^{1/2}$
J	= grouping of coefficients, $\frac{1}{2}(L/D)(I\beta r)^{1/2}$
L	= lift
m	= mass
r	= radial distance from planet center
t	= time
$\bar{u}$	= nondimensional velocity, $V/(gr)^{1/2}$
V	= velocity tangent to the flight path
Y	= nondimensional grouping of parameters and variables, $(\rho_0 g/2B)(r/\beta)^{1/2}\bar{u}f$
$\bar{z}$	= independent variable used in the solution, $\ln(V_e/V)^2$
z_e	= quantity related to the initial velocity, $\ln(V_e^2/gr)$
β	= inverse scale height
γ	= flight path inclination angle
ρ	= atmospheric density

Subscripts and Superscripts

0	= reference condition at sea level
e	= reference condition at initial entry
$(')$	= derivative with respect to $\bar{z}$

Introduction

THE equations governing the motion of a body entering a planetary atmosphere are highly nonlinear, and no general analytical solution to them has been found. Several specialized approximation solutions have been obtained however.

Allen and Eggers[1] obtained an approximate solution to these equations for a ballistic vehicle by assuming that: 1) the gravitational component of the force tangent to the entry trajectory is negligible when compared to the drag force along the path; 2) the inverse scale height β is a constant, which implies an isothermal atmosphere in which the atmospheric density is an exponential function of altitude; and 3) the path inclination angle remains a constant.

The solution, although quite useful, is limited to large entry angles. Eggers, Allen, and Neice[2] extended these results to a lifting vehicle; however, their analysis was mostly concerned with the range of the vehicle.

Norman[3] and others have shown that the effect of the first assumption is small. This fact is especially true at such points of interest as that of maximum deceleration. Work[3, 4] has also been done which shows the effect of the second assumption. The forementioned improvements increase the accuracy of the Allen and Eggers solution, but do not extend its limited range of application.

Chapman[5] studied the problem of entry by numerically solving the equations of motion. The technique used was that of combining the equations into a single equation governing the entry variables, with small quantities being neglected. Boltz[6] extended the work of Chapman to include the effects of the neglected quantities. Eggers[7] also used a single equation describing the relationship of the entry variables to obtain a solution that is useful for analyzing supercircular entry of a skipping vehicle. Loh[8–12] has developed an approximate analytical solution and has shown its agreement over wide regions of lift conditions, entry angles, and entry velocities. However, comparison of Loh's solution with an exact numerical solution shows it to be inaccurate for certain skipping-type trajectories. Many results obtained using Loh's method along with other methods and exact results are given in the book by Loh.[13]

The present work attempts to clarify some of the assumptions required for the derivation of Loh's results. The method of solution developed provides accurate analytical results that are valid over many entry conditions. The solution form is such that the atmospheric density, or altitude, and flight path angle are found as explicit functions of the vehicle velocity.

Development of the Basic Equations

The geometry of a vehicle entering a planetary atmosphere is shown in Fig. 1. From a consideration of the forces acting on the vehicle, the equations of motion may be written as

$$dV/dt = g\sin\gamma - (D/m) \tag{1}$$

Received June 1, 1964; revision received November 9, 1964. This paper is based on a thesis by the second author submitted in partial fulfillment of the requirements for the degree of Master of Science in Astronautics at Purdue University.

* Associate Professor of Aeronautics, Astronautics, and Engineering Sciences; also Consultant, Avco Corporation, Research and Advanced Development Division.

† Graduate Student, School of Aeronautical and Engineering Sciences; now First Lieutenant, U. S. Air Force.

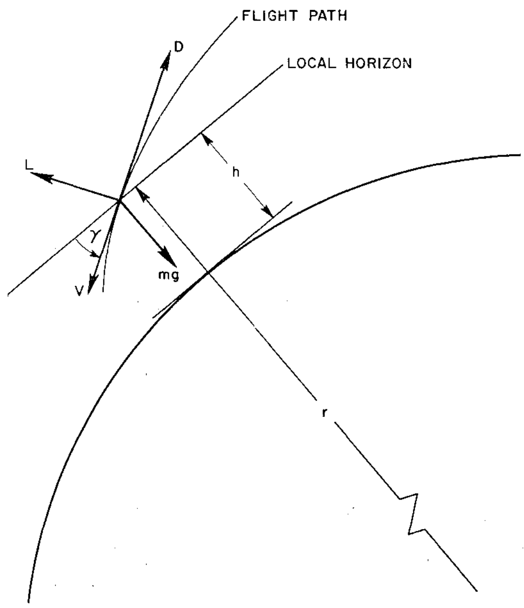

Fig. 1 Entry geometry and nomenclature.

and

$$V(d\gamma/dt) = [g - (V^2/r)]\cos\gamma - (L/m) \qquad (2)$$

In Eq. (1) the acceleration tangent to the flight path is equated to the component of gravitational acceleration acting along the flight path and to the drag deceleration. Equation (2) is the force balance in the direction perpendicular to the flight path, in which the contributions per unit mass of lift, gravity, and centrifugal forces are set equal to the rotational acceleration of the body.

If it is assumed that 1) the acceleration of gravity is constant (r is approximately constant) and 2) C_L and C_D are constant, then Eqs. (1) and (2) are two equations in the three variables velocity, inclination angle, and density. A third relationship is thus needed for their solution. The required equation is

$$d\rho/dt = \beta\rho\, V\sin\gamma \qquad (3)$$

which is derived from the definition of the inverse scale height

$$\beta = -(1/\rho)(d\rho/dh) \qquad (4)$$

and the geometric relation

$$dh/dt = -\,V\sin\gamma \qquad (5)$$

If an isothermal atmosphere is assumed, β is a constant and (4) may be integrated to yield

$$\rho/\rho_0 = e^{-\beta h} \qquad (6)$$

It is also useful to assume

$$g\sin\gamma \ll D/m \qquad (7)$$

which is valid throughout most of the entry trajectory.

Utilizing (7) the equations to be solved become

$$dV/dt = -g\rho V^2/2B \qquad (8)$$

$$V\frac{d\gamma}{dt} = \left(g - \frac{V^2}{r}\right)\cos\gamma - \frac{g\rho V^2}{2B}\left(\frac{C_L}{C_D}\right) \qquad (9)$$

$$d\rho/dt = \beta\rho V\sin\gamma \qquad (10)$$

where $B = mg/C_DA$.

Defining

$$f = \rho/\rho_0 \qquad (11)$$

and following Eggers,[7] Eqs. (8–10) can then be combined into a single equation for the determination of f (density) as a function of velocity. The transformation is accomplished by letting

$$\bar{z} = -\ln(V/V_E)^2 \qquad (12)$$

from which one obtains

$$d\bar{z} = -2(dV/V) \qquad (13)$$

Changing the dependent variable in Eq. (3) from ρ to f and the independent variable from t to $\bar{z}$ one obtains

$$df/d\bar{z} = f' = (\beta B/\rho_0 g)\sin\gamma \qquad (14)$$

Differentiating Eq. (14) with respect to $\bar{z}$ and utilizing Eqs. (8) and (9) one may write

$$f'' = \frac{\bar{I}}{f}\,[\exp(\bar{z} - z_e) - 1] - \bar{J} \qquad (15)$$

where

$$\bar{I} = \beta/r(B\cos\gamma/g\rho_0)^2 \qquad (16)$$

$$\bar{J} = \tfrac{1}{2}(C_L/C_D)(\bar{I}\beta r)^{1/2} \qquad (17)$$

$$z_e = \ln V_e^2/gr = \ln(V_e/V_c)^2 \qquad (18)$$

Since the variation in $\cos\gamma$ is small over the major portion of typical entry trajectories, its value in $\bar{I}$ may be approximated by $\cos\gamma_e$.

Letting

$$I = \beta/r(B\cos\gamma_e/g\rho_0)^2 \qquad (19)$$

Eq. (15) becomes

$$f'' = (I/f)[\exp(z - z_e) - 1] - J \qquad (20)$$

which is seen to be a second order nonlinear differential equation for the determination of f (density) as a function of $\bar{z}$. Since the natural logarithm of f is proportional to h [Eq. (6)], the altitude is thus also known. As can be seen from Eq. (14), the sine of the inclination angle is proportional to the first derivative of f and hence it also is determined as a function of velocity, once f as a function of $\bar{z}$ is known.

Some Previous Solutions

Equation (20) was used by Eggers[7] to obtain an approximate solution useful for studying grazing trajectories. It was also employed by Boltz[6] in his improvement of the work of Chapman.[5] Defining

$$\bar{u} = V/(gr)^{1/2} \qquad (21)$$

$$Y = (\rho_0 g/2B)(r/\beta)^{1/2}\bar{u}f \qquad (22)$$

Equation (20) may be transformed to

$$\bar{u}\frac{d^2Y}{d\bar{u}^2} - \left(\frac{dY}{d\bar{u}} - \frac{Y}{\bar{u}}\right) = \frac{(1-\bar{u}^2)\cos^2\gamma}{\bar{u}Y} - (\beta r)^{1/2}\left(\frac{L}{D}\right)\cos\gamma$$

vert accel	vert component of drag force	g − centrifugal force	lift force

(23)

where the correspondence between left- and right-hand sides of this equation with that of Eq. (20) has been maintained. Equation (23) is identical to Boltz's Eq. (27), which was used for a numerical solution of the variable Y with certain entry conditions.

If the right-hand side of Eq. (20) is neglected, one obtains

$$f'' = 0 \qquad (24)$$

From Eqs. (23) and (20) this is seen to be equivalent to neglecting the gravity and centrifugal forces compared to the inertia and drag forces for a ballistic vehicle. Integrating (24) yields

$$f' = \text{const} = (\beta B/g\rho_0)\sin\gamma_e \qquad (25)$$

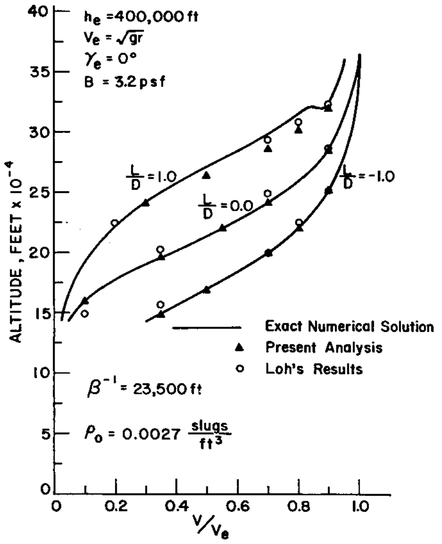

Fig. 2 Comparison of altitude profiles for tangential entry at circular satellite velocity.

where Eq. (4) has been used to fix the constant. Integrating once more, one finds

$$f = [(\beta B/g\rho_0)\sin\gamma_e]\bar{z} + (\rho_e/\rho_0) \tag{26}$$

which can be rewritten as

$$\frac{V}{V_e} = \exp\left\{-\frac{g(\rho - \rho_0)}{2\beta B\sin\gamma_e}\right\} \tag{27}$$

Equation (27) is the result obtained by Allen and Eggers.[1]

Loh[8] departed from the previously mentioned methods to obtain his second order solution. Rearranging Eqs. (1) and (2) under the assumption of an exponential density, he obtained

$$\frac{d}{d\rho}(\cos\gamma) + \frac{1}{\beta r}\frac{\cos\gamma}{\rho}\left(\frac{gr}{V^2} - 1\right) = \frac{1}{2}\frac{L}{D}\frac{g}{\beta B} \tag{28}$$

$$\frac{d}{d\rho}\left(\frac{V^2}{gr}\right) + \frac{g}{\beta B}\frac{V^2/gr}{\sin\gamma} = \frac{2}{\beta r}\frac{1}{\rho} \tag{29}$$

Noting that the particular grouping of variables

$$\frac{1}{\beta r}\left(\frac{\cos\gamma}{\rho}\right)\left(\frac{gr}{V^2} - 1\right) \tag{30}$$

is relatively insensitive to integration over ρ and γ, and thus may be considered constant, Eq. (28) is integrated to give

$$\cos\gamma = \cos\gamma_e + \left[\frac{1}{2}\left(\frac{L}{D}\right)\frac{g}{\beta B} - \frac{1}{\beta r}\frac{\cos\gamma}{\rho}\left(\frac{gr}{V^2} - 1\right)\right](\rho - \rho_e) \tag{31}$$

Neglecting the small right-hand side of Eq. (29), the equation may be rewritten as

$$\frac{d}{d\rho}\left(\frac{V^2}{gr}\right) = -\frac{g\,(V^2/gr)}{\beta B\sin\gamma} \tag{32}$$

Substituting Eq. (28) into (32) and integrating, again considering the grouping of Eq. (30) a constant, yields

$$\ln\left(\frac{V}{V_e}\right)^2 = \frac{(g/\beta B)(\gamma - \gamma_e)}{\frac{1}{2}(L/D)(g/\beta B) - (1/\beta r)(\cos\gamma/\rho)[(gr/V^2) - 1]} \tag{33}$$

which can be rewritten as

$$\frac{g\rho}{\beta B} = \frac{(1/\beta r)[(gr/V^2) - 1]\cos\gamma\,[\ln(V_e/V)^2]}{\frac{1}{2}(L/D)\ln(V_e/V)^2 + (\gamma - \gamma_e)} \tag{34}$$

Combining (31) and (33) to eliminate the denominator of the latter, one obtains

$$\cos\gamma - \cos\gamma_e = \left[\left(\frac{C_DA}{m\beta}\right)\frac{(\gamma - \gamma_e)}{\ln(V/V_e)^2}\right](\rho - \rho_e) \tag{35}$$

Any two of the Eqs. (31, 34, and 35) may be used to determine two of the variables ρ, V, and γ in terms of the third. One notes, however, that since the equations are coupled, an explicit relationship for two of the variables in terms of the third is not obtained. Thus, an iterative numerical technique is required, in general, before actual values for the variables may be obtained.

In the following work, solutions to Eq. (20) will be sought to determine explicitly the density, or altitude, and flight path angle as a function of velocity.

Solution of the Basic Equation

To solve Eq. (20) it was assumed that the quantity f could be represented in a power series of the variable $\bar{z}$. Thus

$$f = f_e + f_e'\bar{z} + \bar{A}\bar{z}^2 + \ldots \tag{36}$$

$$f' = f_e' + 2\bar{A}\bar{z} + \ldots \tag{37}$$

If Eq. (37) is solved for the coefficient $\bar{A}$ and the result substituted into Eq. (36) neglecting terms of higher order than $\bar{A}\bar{z}^2$, one obtains

$$f - f_e = (f' + f_e')(\bar{z}/2) \tag{38}$$

The validity of the approximations made will be verified by a comparison of the approximate solutions obtained with exact results. On the basis of the assumption made in (38), once $f'(\bar{z})$ is known one obtains, without further integration, an expression for f as a function of $\bar{z}$.

A comparison can be made between Eq. (38) and results obtained by Loh who derived Eq. (35) from the equations of motion. It is shown later that Eq. (35) is directly equivalent to Eq. (38), derived here from a series expansion. One might thus infer that Loh's assumption is basically equivalent to the truncation of Eq. (36) after the first three terms.

Multiplying both sides of Eq. (38) by f'' one obtains

$$f'f'' + f_e'f'' = (2/\bar{z})ff''[1 - (f_e/f)] \tag{39}$$

The left-hand side of Eq. (39) may be integrated directly to give

$$\frac{f'^2}{2} + f_e'f' - \frac{3}{2}f_e'^2 = \int_0^{\bar{z}}\left[\frac{2}{\bar{z}}ff''\left(1 - \frac{f_e}{f}\right)\right]d\bar{z} \tag{40}$$

Substituting from Eq. (20) for ff'' on the right-hand side, one obtains

$$f'^2 + 2f_e'f' - 3f_e'^2 = 4IG_1 - 4JG_2 \tag{41}$$

where

$$G_1 = \int_0^{\bar{z}}\left[\left(1 - \frac{f_e}{f}\right)\frac{(e^{\bar{z}-\bar{z}_e} - 1)}{\bar{z}}\right]d\bar{z} \tag{42}$$

$$G_2 = \int_0^{\bar{z}}\left(\frac{f - f_e}{\bar{z}}\right)d\bar{z} \tag{43}$$

The functions G_1 and G_2 are approximately determined in Appendix A to be

$$G_1 = e^{-\bar{z}_e}[E_i^*(\bar{z}) - C - \ln\bar{z}] - (1 - e^{-\bar{z}_e})\ln\left(1 + \frac{f_e'\bar{z}}{f_e}\right) \tag{44}$$

$$G_2 = \tfrac{1}{4}(3f_e' + f')\bar{z} \tag{45}$$

where

$$E_i^*(\bar{z}) = \int_{-\infty}^{\bar{z}}\frac{e^x}{x}dx = C + \ln\bar{z} + \sum_{n=1}^{\infty}\frac{\bar{z}^n}{n\cdot n!}$$

and $C = 0.577\ldots$ (Eulers const). Substituting (44) and (45) into (41), one then obtains

$$f' + 2f_e'f' - 3f_e'^2 = 4I\{e^{-z_e}[E_i^*(\bar{z}) - C - \ln\bar{z}] - (1 - e^{-z_e})\ln[1 + f_e'\bar{z}/f_e]\} - J\bar{z}(3f_e' + f') \quad (46)$$

Equation (46) is quadratic in the variable f' and thus may be solved for f' yielding

$$f' = \left(-f_e' + \frac{J\bar{z}}{2}\right) + 2\,F(\bar{z}) \quad (47)$$

where

$$F(\bar{z}) = \left\{f_e'^2 - \frac{f_e'J\bar{z}}{2} + \left(\frac{J\bar{z}}{4}\right)^2 + I\left[e^{-z_e}(E_i^*(\bar{z}) - C - \ln\bar{z}) - (1 - e^{-z_e}) \times \ln\left(1 + \frac{f_e'\bar{z}}{f_e}\right)\right]\right\}^{1/2} \quad (48)$$

Applying Eq. (38) to (47) one obtains without further integration that

$$f = f_e - (J\bar{z}^2/4) + \bar{z}F(\bar{z}) \quad (49)$$

The quantity f' is known as an explicit function of $\bar{z}$ from Eq. (47). Since the flight path angle is directly proportional to f' from Eq. (14), (47) determines gamma as a function of $\bar{z}$. In a similar manner the determination of f as an explicit function of $\bar{z}$ from Eq. (49) yields density (11) and altitude (6) as functions of $\bar{z}$. One should note that Eqs. (47) and (49) do not require an iterative evaluation as in Loh's work.

Results

In this section the solution developed is compared with other approximate analytic solutions and with exact numerical solutions. Since the solution of Loh appears to be the most accurate over the widest regions of entry conditions, considerable comparison will be made with that work. For the purpose of the comparisons to be made, results of the solution developed will be presented in two groupings. Results for entry at circular satellite velocity will first appear, to be followed by those for super-circular entry.

Entry at Circular Satellite Velocity

The altitude-velocity profiles generated by the solution for various lift conditions are presented in Fig. 2, using entry conditions chosen by Loh. The exact numerical solution to the equations and the approximate results of Loh are also presented for the purpose of comparison. Since the initial entry angle in this figure is zero, these results are especially significant. No previous analytic solution was valid in this region except that presented by Loh. It is interesting to note that over the regions of lift to drag ratio between plus and minus one, both solutions yield good results. The inaccuracy present in both solutions, when the lift to drag ratio is one, can be attributed to the fact that the entry vehicle experiences a slight skip under these conditions. In general, for skipping-type trajectories, the accuracy of both the solution developed by Loh and the one presented here is compromised.

Plots of the variation in the inclination angle during entry under the same initial conditions used in Fig. 2 are shown in Fig. 3. Since the flight path inclination is proportional to the derivative of density, one would expect that any inaccuracy in the previous figure would be magnified in Fig. 3. As can be seen, agreement is still satisfactory.

Figures 4 and 5 are plots for a ballistic vehicle of the altitude profile and flight path angle, respectively, for various entry angles. Figures 2–5 demonstrate the validity of both

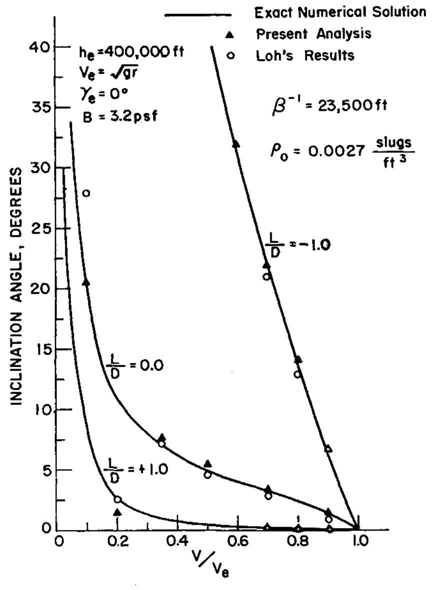

Fig. 3 Comparison of inclination angle variation for tangential entry at circular satellite velocity.

the solution presented and that developed by Loh over wide regions of lifting conditions and initial entry angles. Since both solutions are accurate to the same order of magnitude, it is difficult to judge which provides the better agreement. It is felt, however, that the solution presented here is easier to use in making numerical calculations since an iterative method is not required as in the case of Loh's work.

Entry at Super-Circular Velocity

Figure 6 is a plot of the altitude-velocity profile for ballistic entry at a super-circular velocity as compared with the exact solution of Lovelace[14] and the approximate solutions of Loh and Allen and Eggers. The entry angle chosen is near the value at which the vehicle would pass completely through the atmosphere. The skipping effect shown in the plot may be attributed to the curvature of the planet since the entry vehicle does not develop lift. It is easily seen that for this set of critical entry conditions, neither the solution developed by Loh nor the one presented here provide very good agreement. If an analytical solution is needed for this case, the

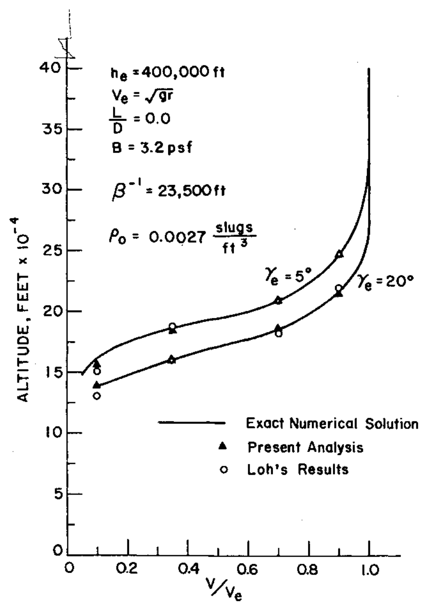

Fig. 4 Comparison of altitude profile for nontangential entry at circular satellite velocity.

simple one developed by Allen and Eggers provides about the same degree of accuracy as any other.

If one increases the initial flight path angle by only a small amount from the critical case described in the previous paragraph, one may see that the solution developed again agrees well with exact results. These results are shown in the altitude-velocity profile of Fig. 7. Although the values generated by the solution developed are more accurate than those of the Allen and Eggers solution, one questions the worth of the somewhat more difficult calculations necessary for altitude determination. One case where such calculations might be justified would be when one wanted the deceleration experienced by the vehicle. Since the deceleration is an exponential function of the altitude, any additional accuracy that one can obtain by an analytic solution is often justified.

Figure 8 again demonstrates the validity of both the solution developed by Loh and that presented here for various ballistic coefficients at super-circular entry velocity. It should be noted that the values calculated from the solution developed agree remarkably well with the exact numerical solution to the equations. It should also be mentioned that the values are slightly more accurate than the values calculated from Loh's solution.

Analytical Comparison with Loh's Solution

The approximate relationship obtained in this work by terminating the series expansion of density is (38)

$$f - f_e = (f' + f_e')(\bar{z}/2) \tag{50}$$

Writing the quantity f and its derivative in terms of the entry variables, and utilizing the definition of $\bar{z}$,

$$(\rho - \rho_e) = -\frac{\beta B}{2g}(\sin\gamma + \sin\gamma_e)\ln\left(\frac{V}{V_e}\right)^2 \tag{51}$$

The approximate series expansion for the sine is employed to yield

$$-(\gamma + \gamma_e)\frac{m\beta}{2C_DA}\ln\left(\frac{V}{V_e}\right)^2 = (\rho - \rho_e) \tag{52}$$

One may multiply both sides of (52) by $(\gamma - \gamma_e)$ and obtain

$$\left(1 - \frac{\gamma^2}{2}\right) - \left(1 - \frac{\gamma_e^2}{2}\right) = \frac{C_DA}{m\beta}\frac{(\gamma - \gamma_e)}{\ln(V/V_e)^2}(\rho - \rho_e) \tag{53}$$

The right-hand side is seen to be the approximate expansion for the cosine, such that

$$\cos\gamma - \cos\gamma_e \simeq \left[\frac{C_DA}{m\beta}\frac{(\gamma - \gamma_e)}{\ln(V/V_e)^2}\right](\rho - \rho_e) \tag{54}$$

which is the relationship obtained in Eq. (35) by combining Loh's results which are given in Eqs. (33) and (34). The result shown in Eq. (54) was obtained by terminating the series expansion of density as a function of velocity, whereas Loh obtained the equivalent result from the observation that the particular grouping of variables shown in Eq. (30) may be considered approximately constant over ρ and γ integrations. The present work thus provides a possible interpretation of Loh's procedure.

Conclusions

From the solution developed here, a possible interpretation has been provided for the statement by Loh that a certain combination of the entry variables is insensitive to ρ or γ integration. This interpretation results from the derivation of the equivalent (54) to one of Loh's two basic equations (33) and (34) by assuming that the atmospheric density can be approximately represented by a power series through the quadratic terms in the logarithm of the velocity.

It should be noted that in every instance examined, the solution presented here provided at least as good agreement with exact numerical solutions as did the solution of Loh. The solution developed is felt to be somewhat easier to use than the solution by Loh as it does not require iterative techniques over any portion of the range of applicability.

Because of the lack of good agreement of both the solution presented and that by Loh for certain skipping-type trajectories, there still does not exist an approximate analytical solution to the entry equations which is valid over all of the entry conditions in which there is interest.

Appendix A: Approximate Determination of the Functions G_1 and G_2

The function G_1, defined by

$$G_1 = \int_0^{\bar{z}}\left[\left(1 - \frac{f_e}{f}\right)\frac{(e^{\bar{z} - z_e} - 1)}{\bar{z}}\right]d\bar{z} \tag{A1}$$

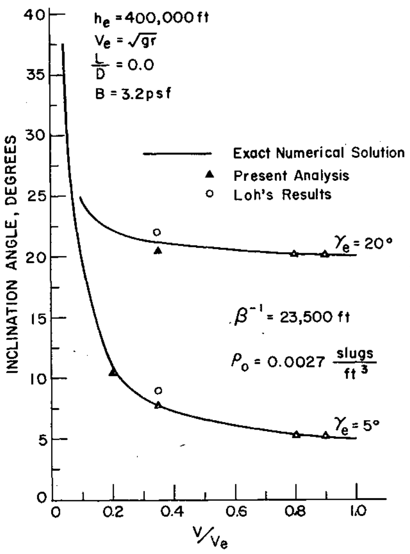

Fig. 5 Comparison of inclination angle variation for nontangential entry at circular satellite velocity.

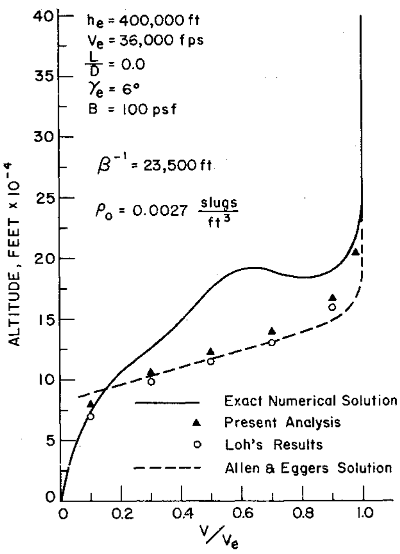

Fig. 6 Comparison of altitude profile for shallow super-circular entry.

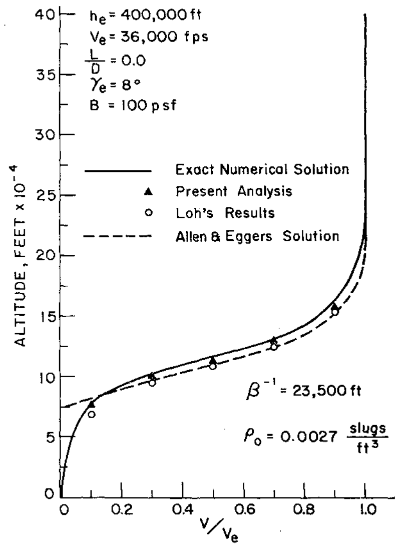

Fig. 7 Comparison of altitude profile for semi-shallow super-circular entry.

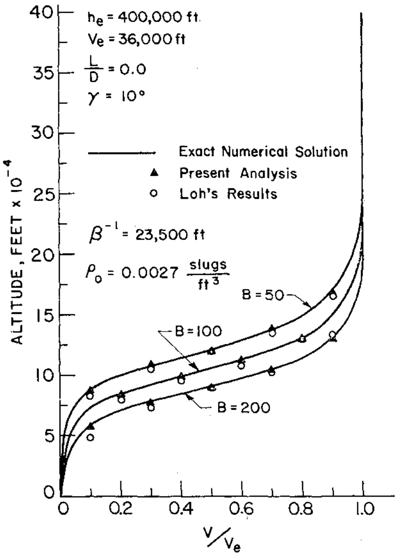

Fig. 8 Effect of ballistic coefficient on altitude profile.

may be approximately determined as shown below. Expanding the integrand yields

$$G_1 = \int_0^{\bar{z}} \left[\frac{e^{\bar{z}-z_e}}{\bar{z}} - \frac{1}{\bar{z}} - \frac{f_e}{f}\frac{e^{\bar{z}-z_e}}{\bar{z}} + \frac{f_e}{f}\frac{1}{\bar{z}}\right] d\bar{z} \quad \text{(A2)}$$

The first two terms on the right-hand side are in a form that may be directly integrated utilizing the series expansion for $\exp\bar{z}$. Since, in general, f_e/f rapidly decreases as $\bar{z}$ increases, the contribution of the second two terms can be expected to be important only for small z. Thus, using

$$e^{\bar{z}} = 1 + \bar{z} + \ldots$$
$$f = f_e + f_e'\bar{z} + \ldots \quad \text{(A3)}$$

and substituting into (A2) one obtains

$$G_1 \simeq \int_0^{\bar{z}} \left\{e^{-z_e}\left[\frac{e^{\bar{z}}}{\bar{z}} - \frac{1}{1 + (f_e'\bar{z}/f_e)}\right] - \frac{1}{\bar{z}} + (1 - e^{-z_e})\frac{1}{\bar{z}[1 + (f_e'\bar{z}/f_e)]}\right\} d\bar{z} \quad \text{(A4)}$$

Each of the terms in (A4) may now be integrated. If small terms are neglected, the result simplifies to yield

$$G_1 = e^{-z_e}\sum_{n=1}^{\infty}\frac{\bar{z}^n}{n \cdot n!} - (1 - e^{-z_e})\ln\left(1 + \frac{f_e'\bar{z}}{f_e}\right) \quad \text{(A5)}$$

Using the definition of the exponential integral in the form

$$E_i^*(\bar{z}) = \int_{-\infty}^{\bar{z}} \frac{e^{\alpha}}{\alpha} d\alpha = C + \ln\bar{z} + \sum_{n=1}^{\infty}\frac{\bar{z}^n}{n \cdot n!} \quad \text{(A6)}$$

where the constant is Eulers const $C = 0.577\ldots$ one need not evaluate the series in (A5), but may instead substitute the relationship of (A6). Values for the exponential integral are tabulated in many sources.[15] Thus, the expression for G_1 is written in its final form

$$G_1 = e^{-z_e}(E_i^*(\bar{z}) - C - \ln\bar{z}) - (1 - e^{-z_e})\ln\left(1 + \frac{f_e'\bar{z}}{f_e}\right) \quad \text{(A7)}$$

By applying the approximation of Eq. (38) to the definition of G_2, where

$$G_2 = \int_0^{\bar{z}} \frac{(f - f_e)}{\bar{z}} d\bar{z} \quad \text{(A8)}$$

one obtains

$$G_2 = \int_0^{\bar{z}} \frac{(f' + f_e')}{2} d\bar{z} \quad \text{(A9)}$$

Integration yields

$$G_2 = \tfrac{1}{2}[f - f_e + f_e'\bar{z}] \quad \text{(A10)}$$

Again applying (38), the expression may be put in the form

$$G_2 = (\bar{z}/4)\,[f' + 3f_e'] \quad \text{(A11)}$$

References

[1] Allen, H. J. and Eggers, A. J., Jr., "A study of the motion and aerodynamic heating of ballistic missiles entering the earth's atmosphere at high supersonic speeds," NACA Rept 1381 (1958).

[2] Eggers, A. J., Jr., Allen, H. J., and Neice, S. E., "A comparative analysis of the performance of long-range hypervelocity vehicles," NACA TN-4046 (1957).

[3] Norman, W. S., "Improvements to the Allen and Eggers solution for ballistic re-entry," U. S. Air Force Rept. USAFA-TN-63-2 (1963).

[4] Meier, T. C., "Improvements to the Allen and Eggers solution for an arbitrary atmosphere," Southwestern Section, AIAA Student Paper Competition, Ft. Worth, Texas (1963).

[5] Chapman, D. R., "An approximate analytical method for studying entry into planetary atmospheres," NACA TR R-11 (1959).

[6] Boltz, F. W., "Z-function solutions for the motion and heating during atmospheric entry from equatorial orbits of a rotating planet," NASA TN D-1555 (1963).

[7] Eggers, A. J., Jr., "The possibility of a safe landing," *Space Technology* (John Wiley & Sons, Inc., New York, 1959) Chap. 13.

[8] Loh, W. H. T., "A second-order theory of entry mechanics into a planetary atmosphere," J. Aerospace Sci. **29**, 1210–1221, 1237 (1962).

[9] Loh, W. H. T., "Super-circular gliding entry," ARS J. **32**, 1398 (1962).

[10] Loh, W. H. T., "Author's reply to comment by Kenneth Wang," AIAA J. **1**, 978 (1963).

[11] Loh, W. H. T., "Some exact analytical solutions of planetary entry," AIAA J. **1**, 836–842 (1963).

[12] Loh, W. H. T., "Orbital re-entry at small lift-drag ratio and small initial inclination angles," ARS J. **32**, 779 (1962).

[13] Loh, W. H. T., *Dynamics and Thermodynamics of Planetary Entry* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963).

[14] Lovelace, U. M., "Charts depicting kinematic and heating parameters for a ballistic re-entry at speeds of 26,000 to 45,000 feet per second," NASA TN D-968 (1961).

[15] Jahnke, E., Emde, F., and Losch, F., *Tables of Higher Functions* (McGraw-Hill Book Co., Inc., New York, 1960).